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Power Law Growth for the Resistance in the Fibonacci Model

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Many one-dimensional quasiperiodic systems based on the Fibonacci rule, such as the tight-binding Hamiltonian $H\psi(n)=\psi(n+1)+\psi(n-1)+\lambda v(n)\,\psi(n)$, $n\in\mathbb{Z}$, $\psi\in l^2(\mathbb{Z})$, $\lambda\in\mathbb{R}$, where $v(n)=[(n+1)\,\alpha]-[n\alpha]$, $[x]$ denoting the integer part of x and α the golden mean $(\sqrt{5}-1)/2$, give rise to the same recursion relation for the transfer matrices. It is proved that the wave functions and the norm of transfer matrices are polynomially bounded (critical regime) if and only if the energy is in the spectrum of the Hamiltonian. This solves a conjecture of Kohmoto and Sutherland on the power-law growth of the resistance in a one-dimensional quasicrystal.

KEY WORDS: Periodic Hamiltonian; Fibonacci chain; transfer matrix.

1. INTRODUCTION

Quasiperiodic structures play an important role in solid state physics,[10,34] based on the experimental evidence of quasicrystals.[31] In quantum mechanics, it is important to know when the states are localized, extended, or critical, depending on the value of the energy. We characterize here one aspect of this critical behavior for the following two-valued potential, which has been intensively studied:

$$(H\psi)(n)=\psi(n+1)+\psi(n-1)+\lambda\vartheta(n)\,\psi(n),\qquad n\in\mathbb{Z} \tag{1}$$

where $\vartheta(n)=[(n+1)\,\alpha]-[n\alpha]$, $[x]$ denoting the integer part of x, $\psi\in l^2(\mathbb{Z})$, $\lambda\in\mathbb{R}$, and α is the golden mean $(\sqrt{5}-1)/2$.[3–5,13,14,18,26,30] The

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spectrum of this Hamiltonian is a Cantor set of zero Lebesgue measure for $\lambda \neq 0$ even if α is any irrational number. Moreover, it is singular continuous.[3,30] The essential tool for these studies is the transfer matrix technique:

Let ψ be a solution of $H\psi = E\psi$ and $T(n)$ be the transfer matrix

$$T(n)\begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix} = \begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix}, \qquad 1 \leqslant n \in \mathbb{N}$$

namely

$$T(n) = \begin{pmatrix} E - \vartheta(n) & -1 \\ 1 & 0 \end{pmatrix}$$

Define

$$M(n) = T(n) \cdots T(2)\, T(1) \in SL(2, \mathbb{R})$$
$$M_n = M(q_n)$$

where q_n are the Fibonacci numbers given by the recursion rule

$$q_{n+1} = \quad q_n + q_{n-1} \qquad \text{with} \quad q_0 = 1, \quad q_1 = 1$$

These numbers are associated to the golden mean $\alpha = (\sqrt{5} - 1)/2$ since q_{n-1}/q_n are the best rational approximants of α ($|\alpha - q_{n-1}/q_n| < 1/q_n q_{n+1}$).

It is easy to compute the spectrum of H because it is related to the trace of transfer matrices. Actually, for the tight-binding model (1), one has the recursion

$$M_{n+1} = M_{n-1} M_n \tag{2}$$

Naturally M_n depends on the energy E. Equation (2) implies

$$x_{n+1} = x_n x_{n-1} - x_{n-2}, \qquad x_n = \mathrm{trace}(M_n) \tag{3}$$

All physical information concerning the system can be extracted from (2). Conversely, there exist many different properties associated with interesting physical models, for instance, the electronic and phonon properties for tight-binding interactions[12] and the optical properties of multilayers.[16] Kohmoto and Sutherland,[11,32] taking the Landauer formula as a working definition of resistance, have calculated the electrical resistance of a sample. They conjectured that, for a one-dimensional noninteracting electronic system which contains n scattering potentials, the resistance $\rho(n)$ is bounded by a power of the sample length n if and only if the energy is in the dynamical spectrum, i.e., if $(x_n)_n$ is a bounded sequence.[5,14]

This note is devoted to a proof of this conjecture. As indicated before, the only tool is the recursion relation (2), so the results apply as well to different situations. We emphasize only few examples.

Electronic case:

$$t_{n+1}\psi_{n+1}+t_n\psi_n=E\psi_n$$

where ψ is the wave function and $(t_n)_n$ is a Fibonacci sequence with two kinds of hopping terms.

Phonon case:

$$t_{n+1}\psi_{n+1}+t_n\psi_{n-1}-(t_{n+1}+t_n)\,\psi_n=-\omega^2\psi_n$$

where ψ denotes the displacement from the equilibrium position and $(t_n)_n$ is a Fibonacci sequence of two spring constants.

Kronig–Penney model on a Fibonacci lattice with Hamiltonian

$$H=-\frac{d^2}{dx^2}+\lambda\sum_j \delta(x-x_j)$$

where $\lambda>0$ and x_j is a Fibonacci sequence.[1,12,15,33,37]

Sütő[29,30] proved that the Lyapunov exponent

$$\gamma(E)=\lim_n\left(\frac{\log\|M_n(E)\|}{n}\right)$$

is zero when E is in the spectrum of H. We improve his result by showing that $\|M(n,E)\|$ is polynomially bounded if and only if E is in the spectrum of H.

2. THE RESULT

Let $\sigma(H)$ be the spectrum of H.

Theorem 1. $E\in\sigma(H)$ if and only if $\|M(n)\|$ is polynomially bounded in n: There exists a constant β such that $\|M(n)\|\leqslant n^\beta$, $\forall n$, $1\leqslant n\in\mathbb{N}$.

Corollary 2. $E\in\sigma(H)$ if and only if $\exists\beta$ such that $|\psi(n)|\leqslant|n|^\beta$, $\forall n\in\mathbb{Z}$, $n\neq 0$.

In this model, the power law growth is a consequence of the equality of the spectrum and the dynamical spectrum. So it is not clear that the corollary still applies in more general situations.

For instance, the classical theory of eigenvalue expansions for Carleman operators (see ref. 2, Chapter 5) implies, in a very general situation (bounded potential) that for almost all spectral values with respect to the spectral measure and for $\varepsilon > 0$, there exist generalized eigenfunctions which grow not faster than $|n|^{1+\varepsilon}$. Here, the results are valid for all spectral values and the price to be paid is the dependence of β on λ. Clearly our estimates are not optimal (think of the case $\lambda = 0$), but improvement on the real physical dependence of the resistance on the coupling constant by numerical computation appears as a difficult question: Recall that the spectrum is singular continuous.

Proof. By definition,

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = M(n) \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}, \qquad n \geqslant 1$$

The symmetry of the potential around $-1/2$ gives

$$\begin{pmatrix} \psi(-(n+1)) \\ \psi(-(n+2)) \end{pmatrix} = L(n) \begin{pmatrix} \psi(-1) \\ \psi(-2) \end{pmatrix}, \qquad n \geqslant 1$$

where $L(n) = T(n)^{-1} \cdots T(2)^{-1}\, T(1)^{-1}$.[29] The analysis of the behavior of $L(n)$ is similar to that of $M(n)$ and the corollary is easily verified.

Corollary 3. The electrical resistance of a Fibonacci chain of scatterers is polynomially bounded in the number of scatterers if and only if the energy is in the spectrum.

Proof. Due to Landauer's formula, the resistance $\rho(n)$ for n scatterers is given by $\rho(n) = \frac{1}{4}(\|M(n)\|_2^2 - 2)$, where $\|A\|_2^2 = \mathrm{Trace}(A^*A)$.[16] Since $\|A\| \leqslant \|A\|_2 \leqslant \sqrt{2}\,\|A\|$ for any $A \in SL(2, \mathbb{R})$, Theorem 1 can be applied.

We will use freely Sütő's result:

Lemma 4. (i) $E \in \sigma(H)$ if and only if $c = \sup_n |x_n| < \infty$.

(ii) Let $E \in \sigma(H)$. Then there exists a constant $a \geqslant \max(c, 2)$ such that

$$\|M_n\| \leqslant a^n, \qquad \forall n,\ 1 \leqslant n \in \mathbb{N} \tag{4}$$

For the sake of completeness and since it is very simple, let us indicate that (4) follows by induction from

$$M_n = M_{n-2} M_{n-1} = M_{n-2}(x_{n-1} \mathbb{1} - M_{n-1}^{-1}) = x_{n-1} M_{n-2} - M_{n-3}^{-1}$$

with

$$\|M_{n-3}^{-1}\| = \|M_{n-3}\|$$

Notation. We denote by $P_k(y_1,..., y_n)$ any polynomial of degree less than k of the variables $(y_1,..., y_n)$. For $P_k(x)=\sum_{j=1}^{k} \alpha_j x^j$ we introduce

$$|P_k|\,(x)=\sum_{j=1}^{k} |\alpha_j|\, x^j$$

with the natural generalization in many variables. The key point toward the proof of Theorem 1 is the linearization of a product $M_n M_m$ with a control on the growth of the coefficients:

Lemma 5. Let $E\in\sigma(H)$. Let $n, k\in\mathbb{N}$ with $n\geqslant 2$. Then there exist four polynomials $P_k^{(i)}(x_{n-1}, x_n,..., x_{n+k})$, $i\in\{1, 2, 3, 4\}$, such that

$$\begin{gathered} M_n M_{n+k}=P_k^{(1)} M_{n+k}+P_k^{(2)} M_{n+k-1}+P_k^{(3)} M_{n+k-2}+P_k^{(4)}\mathbb{1} \\ \sum_{i=1}^{4} |P_k^{(i)}|\,(|x_{n-1}|,..., |x_{n+k}|)\leqslant (2c+1)^k \end{gathered} \tag{5}$$

Lemma 6. Let $n\in\mathbb{N}$. Then

$$M_n M_{n+2}=x_n M_{n+2}-M_{n+1} \tag{6}$$

$$M_n M_{n+3}=x_{n+2} M_{n+2}-\mathbb{1} \tag{7}$$

$$M_n=x_{n-1} M_{n-2}+M_{n-3}-x_{n-3}\mathbb{1} \tag{8}$$

Proof.

$$M_n M_{n+2}=M_n^2 M_{n+1}=(x_n M_n-\mathbb{1})\, M_{n+1}=x_n M_{n+2}-M_{n+1}$$

$$M_n M_{n+3}=M_n M_{n+1} M_{n+2}=M_{n+2}^2=x_{n+2} M_{n+2}-\mathbb{1}$$

$$\begin{aligned} M_{n-3} &= M_{n-3} M_{n-1} M_{n-1}^{-1} \\ &= (x_{n-3} M_{n-1}-M_{n-2})\, M_{n-1}^{-1} \qquad \text{by (6)} \\ &= x_{n-3}\mathbb{1}-M_{n-2}(x_{n-1}\mathbb{1}-M_{n-1}) \\ &= x_{n-3}\mathbb{1}-x_{n-1} M_{n-2}+M_n \end{aligned}$$

Proof of Lemma 5. If $k\in\{0, 1\}$, then (5) follows from $M_n^2=x_n M_{n-1}-\mathbb{1}$ and (8). Moreover,

$$\sum_{i=1}^{4} |P_1^{(i)}|\,(|x_{n-1}|,..., |x_n|, |x_{n+1}|)\leqslant 2c+1$$

Assume the lemma for $k \in \{1,..., l\}$. We have

$$\begin{aligned}
M_n M_{n+l+1} &= (M_n M_{n+l-1})\, M_{n+l} \\
&= (P^{(1)}_{l-1} M_{n+l-1} + P^{(2)}_{l-1} M_{n+l-2} \\
&\quad + P^{(3)}_{l-1} M_{n+l-3} + P^{(4)}_{l-1} \mathbb{1})\, M_{n+l} \\
&= P^{(1)}_{l-1} M_{n+l+1} + P^{(2)}_{l-1}(x_{n+l-2} M_{n+l} - M_{n+l-1}) \\
&\quad + P^{(3)}_{l-1}(x_{n+l-1} M_{n+l-1} - \mathbb{1}) + P^{(4)}_{l-1} M_{n+l}
\end{aligned}$$

To simplify the notations, we neglected the variables $(|x_{n-1}|,..., |x_{n+l-1}|)$. Defining

$$\begin{aligned}
P^{(1)}_{l+1} &= P^{(1)}_{l-1} \\
P^{(2)}_{l+1} &= x_{n+l-2} P^{(2)}_{l-1} + P^{(4)}_{l-1} \\
P^{(3)}_{l+1} &= -P^{(2)}_{l-1} + x_{n+l-1} P^{(3)}_{l-1} \\
P^{(4)}_{l+1} &= -P^{(3)}_{l-1}
\end{aligned}$$

we get polynomials of order less than $l+1$ and

$$\sum_{i=1}^{4} |P^{(i)}_{l+1}| \leqslant (c+1) \sum_{i=1}^{4} |P^{(i)}_{l-1}|$$

So the lemma is true at the order $l+1$.

Proof of Theorem 1. Let $n \in \mathbb{N}^*$. There exists a unique index set of integers n_l, $l \in \{0,..., N\}$, such that $n_{l+1} - n_l \geqslant 2$ and $n = \sum_{l=0}^{N} q_{n_l}$. Since $\vartheta(k+q_l) = \vartheta(k)$ if $1 \leqslant k < q_{l+1} - 1$,[29] then

$$\begin{aligned}
M(n) &= T\left(\sum_{l=0}^{N-1} q_{n_l} + q_{n_N}\right) \cdots T(q_{n_N}) \cdots T(1) \\
&= T\left(\sum_{l=0}^{N-1} q_{n_l}\right) \cdots M_{n_N} \\
&= M_{n_0} M_{n_1} \cdots M_{n_N}
\end{aligned}$$

Suppose first that $E \in \sigma(H)$ and let us prove recursively on $k \geqslant 1$ that

$$\|M_{n_0} M_{n_1} \cdots M_{n_k}\| \leqslant b^{-n_0 + 2(k-1)} (ab)^{n_k} \tag{9}$$

with $b = 2c+1$.

Let us check the case $k=1$: By Lemmas 4 and 5,

$$\begin{aligned}\|M_{n_0}M_{n_1}\| &\leqslant |P^{(1)}_{n_1-n_0}|\ \|M_{n_1}\| + |P^{(2)}_{n_1-n_0}|\ \|M_{n_1-1}\| \\ &\quad + |P^{(3)}_{n_1-n_0}|\ \|M_{n_1-2}\| + |P^{(4)}_{n_1-n_0}| \\ &\leqslant \sum_{i=1}^{4} |P^{(i)}_{n_1-n_0}|\ a^{n_1} \leqslant b^{n_1-n_0}a^{n_1}\end{aligned}$$

Assume (9) for $k\in\{1,..., l\}$. Then using Lemma 5 again, we have

$$\begin{aligned}\|M_{n_0}M_{n_1}\cdots M_{n_l}M_{n_{l+1}}\| &= (M_{n_0}M_{n_1})\,M_{n_2}\cdots M_{n_{l+1}}\| \\ &\leqslant |P^{(1)}_{n_1-n_0}|\ \|M_{n_1}M_{n_2}\cdots M_{n_{l+1}}\| \\ &\quad + |P^{(2)}_{n_1-n_0}|\ \|M_{n_1-1}M_{n_2}\cdots M_{n_{l+1}}\| \\ &\quad + |P^{(3)}_{n_1-n_0}|\ \|M_{n_1-2}M_{n_2}\cdots M_{n_{l+1}}\| \\ &\quad + |P^{(4)}_{n_1-n_0}|\ \|M_{n_2}\cdots M_{n_{l+1}}\| \\ &\leqslant |P^{(1)}_{n_1-n_0}|\ b^{-n_1+2(l-1)}(ab)^{n_{l+1}} \\ &\quad + |P^{(2)}_{n_1-n_0}|\ b^{-n_1+1+2(l-1)}(ab)^{n_{l+1}} \\ &\quad + |P^{(3)}_{n_1-n_0}|\ b^{-n_1+2+2(l-1)}(ab)^{n_{l+1}} \\ &\quad + |P^{(4)}_{n_1-n_0}|\ b^{-n_2+2(l-2)}(ab)^{n_{l+1}} \\ &\leqslant b^{-n_1+2l}(ab)^{n_{l+1}} \sum_{i=1}^{4} |P^{(i)}_{n_1-n_0}| \\ &\leqslant b^{-n_0+2[(l+1)-1]}(ab)^{n_{l+1}}\end{aligned}$$

and (9) is true for $k=l+1$.

This yields

$$\|M(n)\| = \|M_{n_0}\cdots M_{n_N}\| \leqslant b^{2N}(ab)^{n_N}$$

Since $n_{l+1}-n_l\geqslant 2$, $n_N-n_0\geqslant 2N$ and $\|M(n)\|\leqslant d^{n_N}$ with $d=ab^2$. Moreover,

$$q_k=\frac{1}{\sqrt{5}}\,[\alpha^{-k}+(-1)^{k+1}\,\alpha^{+k}] \quad\text{and}\quad \lim_{k\to\infty}\left|k-\frac{\log(\sqrt{5}\,q_k)}{\log\alpha^{-1}}\right|=0$$

Thus for N large enough,

$$\|M(n)\| \leqslant d^{[\log(\sqrt{5}\,q_k)]/\log\alpha^{-1}} \leqslant (d^{(\log\sqrt{5})/\log\alpha^{-1}})^{\log n}$$

which proves the part "only if" of Theorem 1 with $\beta=(\log\sqrt{5}\log d)/\log\alpha^{-1}$.

Conversely, let E be real such that $\|M(n)\| \leqslant n^{\beta}$ when n goes to infinity. Since $|\text{Trace}\ (A)| \leqslant 2\ \|A\|$ for any $A \in M_{2\times 2}(\mathbb{R})$, $|x_n| \leqslant 2q_n^{\beta}$. But if $E \notin \sigma(H)$, there exist a constant f and $N \in \mathbb{N}$ such that $|x_n| \geqslant f^{q_n}$, $\forall n \geqslant N$,[29] so a contradiction, and $E \in \sigma(H)$.

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